Actuarial Science-A Survey of Theoretical Development

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ACTUARIAL SCIENCE—A SURVEY OF THEORETICAL DEVELOPMENTS*

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A CTUARIAL SCIENCE, at least as the term is used in the English speaking countries, differs radically from some other sciences in that great amounts of practical material are included. For this reason, a survey of the field, such as we customarily attempt at the turn of centuries and half centuries, is most effectively made in two articles, one on theory, one on practice. Conceive, if you please, that botany, horticulture, and greenhouse operation were all lumped together under the heading Plant Science, and you would have something of the same situation.

Here we are concerned with theoretical developments and it is just here that we come up against the first problem. While it is true that no sharp line of demarcation exists between theory and practice, it is not too difficult to make a rough separation of the material on empirical grounds. This, however, does not provide a categorical answer to the question of just what, fundamentally, constitutes a theoretical development of Actuarial Science. Before attempting to answer this question, a brief review of the subject matter is useful in order to provide a point of departure.

There is probably no better method of summarizing the content of actuarial theory than to list and comment on the subjects studied by beginners in the profession in the order that they are taken up in their course of study. After passing tests to assure himself of the adequacy of his formal mathematical equipment, which includes some training in finite differences and the theory of probability and mathematical statistics, the student at once plunges into the basic insurance mathematics generally known by the term "life contingencies." This subject presupposes the existence of a mortality table, which might be thought of as sprung fully computed from the head of Jove; it has been defined by the famous English actuary, George King, as "the instrument by means of which are measured the probabilities of life and the probabilities of death." From this table are derived the net premiums and reserves for the various forms of life insurance policies as well as the expectation of life and the solution to population problems. The student is then led into the intricacies of working with multiple decrement tables, which enable him to handle the effect of sickness, withdrawal,

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etc., along with mortality. Finally, he must learn the mathematical methods of computing tables, given the basic figures.

The remainder of the course has to do with the way mortality tables actually come into being—how they may be constructed from population data or from the records of insurance companies. Here there are two problems involved. First, there is the extraction of a set of mortality rates from the data. The resulting series of figures usually shows quite an irregular progression, which is at variance with our ideas of what a mortality table should be. The remaining task is to substitute for this sequence a smoothly progressing series without doing too much violence to what might be termed the "indications" of the observed series. The term "graduation" is applied to this process, and a great deal has been written on the subject both in actuarial journals and elsewhere. Naturally, both of these processes may be applied to other matters than mortality rates, for example, to sickness rates, accident rates, etc.; and graduation covers the smoothing of series arising in fields far removed from insurance and demography.

In European countries, the university courses go on to more abstruse matters including the theory of risk and the use of advanced mathematical techniques. In the English speaking countries, where university courses do not usually progress beyond the elementary parts of the subject and the actuarial student is pursuing his studies "on the job," the advanced topics are left to individuals with a special bent.

WHAT ARE THEORETICAL DEVELOPMENTS?

Here, then, is the field of actuarial science in very brief outline, stopping short of the practical parts of it. It is too much to expect developments of a mathematical nature in life contingencies any more than one would look for new material in a course of elementary mechanics. George King put it this way in the preface to his text book written for the British Institute of Actuaries and published in 1887: "This volume, from the nature of the case, includes but little that is actually new in the way of investigation." Of course, the arithmetic of the newer policies and of more recent valuation methods has had to be worked out, but this has presented few difficulties. The foundations of the subject are another matter. Since the theory of probability has been revolutionized, it has become imperative to reexamine in the light of the new techniques of this theory all the results flowing from the basic assumption of as ancient a piece of baggage as the mortality table.

Returning now to the question proposed—what constitutes a theoretical development—it becomes evident that there are two general

fields which can produce such developments. The first is purely mathematical—the creation of mathematical processes to implement actuarial ideas as well as the application of existing methods which have not been used in the actuarial field but have proved their worth elsewhere. The other is the examination of the axioms and fundamentals of the actuarial processes so as to be assured that the real meaning of what actuaries are doing is what they believe it to be, or, failing this, to establish the limits wherein the traditional actuarial formulas are valid. Much more time has been spent on the first type of development. As the subject is mathematical, almost algebraic, in nature, the results can be described and evaluated with precision, a procedure which is impossible in the other field. What better reason could one want for making a start with the strictly mathematical developments? These have been most extensive in the fields of graduation and interpolation.

MATHEMATICAL DEVELOPMENTS

At the turn of the century, which is a good reference point for a mid-century survey, there were, in addition to procedures based on interpolation, three methods of graduating mortality tables in general use. These three methods will be described first, saving the interpolation methods until later.

Early Graduation Methods

The basic method of graduation is the free-hand drawing of a curve among points representing the observed rates of death, charted according to age. This graphic method is of course still in use and presumably always will be. The only significant improvement that has been devised is the addition above and below each charted point of two points distant by some convenient multiple of the standard deviation of the ordinate, considered as a frequency variate. These points serve as a guide to the reliability of the observations and to how much departure from them can reasonably be ascribed to chance fluctuations. For example, if the "convenient multiple" is $\frac{2}{3}$, one would expect about half of the graduated points to fall within the intervals formed by the pairs of guide points.

Another popular method of graduation is peculiar to life insurance. It depends on Makeham's Law, which states that in many mortality experiences the force of mortality, μ_x at age x, is made up of a constant plus a geometric progression, viz.,

$$\mu_x = A + Bc^x,$$

where A, B and c are constants to be determined. Although most

mortality experiences exhibit some deviation from this type of relationship, a certain amount of variation has been condoned because a mortality law of this type results in a material simplification of the arithmetic of joint-life premiums and reserves, amounting, in the most important field, to the use of a single entry table in place of a whole series of double entry tables.

The problem of finding the most satisfactory methods of fitting a curve of this type to the data was a major concern of actuaries at the turn of the century. Once the exponential constant c was known, the others could be determined by the method of least squares. Various methods of finding c were explored; oddly enough, it was not until the mid-thirties that anyone had enough wit to graph μ_x minus various constants on logarithmic paper and determine from the graphs the geometric progression, i.e., the curve nearest to a straight line. Recent years have produced methods of modifying the constants in different parts of the series in such a manner as to fit the unadjusted data more closely and yet preserve the arithmetical simplifications referred to.

In the third of the classical graduation methods, the adjusted or graduated term of a series is a linear compound of a fixed number of unadjusted terms among which it is centrally located. A symmetrical series of weights constitutes the array of coefficients of the linear compound; both the weights and the number of terms involved may be varied to produce a whole family of formulas. Some graduation formulas achieve smoothness by riding roughshod over the irregularities of the data; others are relatively faithful to the progression of the data at the expense of smoothness. The more terms in the linear compound, the smoother the graduation formula can be made. There is no best formula; one that will produce fine results when used on one series of data may show poor ones on another.

Originally, the only formulas of this family which were considered practical were the ones which involved simple arithmetical operations. These were known as "summation formulas," since most of the arithmetic consisted of the summation of groups of terms, somewhat like a complicated variety of moving average. Since the general adoption of calculating machines this limitation is no longer necessary. The question of which came first, the machine or the machine method is a fascinating one. The linear-compound graduation formula which produces the smoothest results when judged by the reduction of error in the third differences has been discovered independently no less than four times, in 1871, 1915, 1916, and 1918. DeForest's original discovery is obviously pre-machine, and Larus's in 1918 definitely post-machine, since it was accompanied by a detailed description as to how to do the

work on a machine. The other two are difficult to judge. Perhaps what happens at an intermediate time is that the machine is getting just common enough to promote rather than rule out an investigation which appears to involve considerable arithmetical detail.

The difference-equation method of graduation was developed during such a transition period. Whittaker's original paper was read in 1919 in Scotland, where there is a dearth of calculating machines. The practical application dates from 1924 when Henderson's first paper was published in this country, where there are a great many of them. Approximations to the practical results of the new method but using the summation technique appeared in Scotland in 1926, a curious procedure which was evidently designed to secure the advantage of the method while bypassing the arithmetic. But enough of the philosophy of the machine! A really new method of graduation calls for a short description even in a brief survey like this, particularly since this one gets back to fundamental principles.

Difference-Equation Graduation

The point of departure of the new method is to establish measures of the opposing forces of graduation: smoothness and closeness of fit. Assuming that adequate measures can be found, it is possible to solve the graduation problem by maximum and minimum methods: by finding the smoothest graduated series compatible with a certain degree of closeness of fit. The usual measure of smoothness, S, is the sum of the squares of some order of differences of the n unknown graduated values: $u_1, u_2, \cdots, u_n$. Thus if third differences are used,

$$S = \sum_{x=1}^{n-3} (u_{x+3} - 3u_{x+2} + 3u_{x+1} - u_x)^2.$$

The customary measure of closeness of fit, F, is the sum of the squares of the departures of the n graduated values, u_i, from the n ungraduated values, v_i, sometimes weighted as in least-squares formulations. Thus if w_i are the weights,

$$F = \sum_{x=1}^{n} w_x(u_x - v_x)^2.$$

Now, if ϵ is a relative measure of the compromise between smoothness and closeness of fit, the expression to be made a minimum is $S + \epsilon F$, and the necessary conditions for a solution come from setting the n partial derivatives of this expression with respect to each of the

n unknowns equal to zero. The result of this differentiation is a set of n linear equations in the n unknown graduated values. Such a set may be solved by machine methods but it is still a long job. If n is large, and it is often between 50 and 100, hand methods are completely inadequate.

If the variation in the weights of the different observations may be ignored, a marked simplification results. Most of the linear equations then follow the same pattern, and thus may be represented by a linear difference equation with constant coefficients. This equation proves to be factorable into two of lower degree, and the arithmetical solution is thereby greatly facilitated.

During the past twenty years the mechanics of the new method have been worked out in detail. There has also been considerable study of the theory. A machine has even been constructed, of springs and wires, which will make a graduation by this method in several special cases. Moreover, one particular case, it turns out, involves such simple arithmetic that the graduation can be performed comfortably without a machine.

The only other new graduation method developed in recent years is a Scandinavian invention which depends on building up the graduated series from an array of second derivatives. This method depends to a great extent on the skill of the man making the graduation, and two independent graduations of the same data might result in a wide variation. In this respect it stands at the other extreme from the Whittaker-Henderson process, which, once the end conditions have been set and the S, F and ϵ criteria established, rolls on to an invariable result.

Osculatory Interpolation

The methods of graduation just described generally involve the substitution of a smooth series of values for an unadjusted series of the same length. It frequently happens in actuarial work that the unadjusted series consists of, say, quinquennial values. In these cases, what is required is a series of interpolated values—four to each interval. The usual interpolation methods of the calculus of finite differences were originally developed for use with smooth analytic functions—such as logarithmic or trigonometric tables. When they are applied to more irregular data, the results are often not as smooth as could be desired, because the arcs that span the intervals between plotted data do not join evenly with one another, but have different tangents at the junction points. To procure a style of interpolation that came closer to fitting actuarial needs, Sprague, back in 1888, constrained the in-

terpolated arcs to join each other at a fixed angle and a specified curvature based on the conventional curves, and the process became known as osculatory interpolation. At the beginning of the century, Karup produced a simpler formula by omitting the matching curvature requirement. Work in this field has been active during succeeding years. The use of specified tangents and curvatures was abandoned, producing more flexible results. We find Henderson writing, in 1924, "I prefer, however, to look upon osculatory interpolation as an entirely self-supporting operation not depending in any way on the theorems of ordinary interpolation. The successive intervals are filled in by curves of the specified degree but the constants are not determined by equating the differential coefficients at the points of junction to those determined from the usual finite-difference interpolation formula. They are simply determined so that the coefficients in the two curves meeting at that point shall be equal to one another. In this way it is not necessary to use a curve of so high a degree in order to secure osculation of a given order."

A further modification of the theory was due to Jenkins, who produced interpolating curves in 1926 which although preserving tangency and curvature at junction points failed to duplicate the original values at these points. In this way, a certain amount of graduation could be combined with the interpolation process. This novel procedure was christened "modified osculatory interpolation." The general theory covering both types was expounded by Greville in 1944 in such detail as to sum up the subject definitely and probably give future research in this field the character of a peroration.

Other Interpolation Methods

During the past ten years, as the gaps in the theory of osculatory interpolation were being filled in, students found that curves produced by this method often exhibited a sort of groundswell, from interval to interval, of such magnitude as to be objectionable. Other methods of interpolation were sought. One line of investigation was to use the Whittaker-Henderson approach and require the interpolated values to be the set with minimum differences of a specified order, as measured by the sum of their squares. This led to a difference equation and produced interpolated values each contributed to by *all* the given values. Greville's elaborate description of this technique appeared in Brazil, in Portuguese, in 1946.

Meanwhile Beers accomplished something of the same thing without resorting to the rather cumbersome difference-equation technique. He

sought a similar set of interpolated values but one in which each would be a linear compound of just the nearest six of the given values. It turned out that this problem could not be solved without making some assumptions in regard to what might be termed the degree of haphazardness to be expected in the series of differences of the order minimized, but the resulting formulas gave eminently satisfactory results.

Both of these methods have been adapted to the production of series of interpolated values of the "modified" type, which does not duplicate the given values and thus involves an element of graduation.

Recently there have been some other interesting theoretical investigations. Aubrey White has developed a finite-difference analogue of osculatory interpolation based on the equality of certain orders of sub-differences, rather than derivatives, at the junction points. Several writers have traced the connection between interpolation formulas and the summation formulas of graduation. There have been some notable contributions to ordinary interpolation: we have Aitken's elegant process of interpolation by "cross-means" and the so-called "throwback" device of E. W. Brown, rediscovered by Camp and Comrie in 1928. Since this is to be a brief survey, we may not dwell further on this topic or describe these interesting developments, but this is as good a place as any to refer to the available actuarial literature. A complete bibliography is out of the question as an appendage to this survey: the tail would be vastly bigger than the dog and imperil its equilibrium. Nearly all of the original articles may be found in the actuarial journals, principally the *Transactions* and the *Record* of the two American societies now merged into the Society of Actuaries; the *Journal* of the British Institute, and the *Transactions* of the Scottish Faculty. Developments in graduation and interpolation up to about 1940 are conveniently listed in Wolfenden's book on mathematical statistics; they are brought down to date in Greville's paper in the September 1948 issue of the *Journal of the American Statistical Association*. A useful elementary work is Morton Miller's *Elements of Graduation*, published by the Society of Actuaries.

Miscellaneous Matters

Prominent in the American actuarial literature of recent years is a series of papers on the derivation of mortality rates from the records of life insurance companies. The arithmetic of multiple decrement tables has been elaborated; the necessary approximations arising from the use of the force rather than the rate of mortality have been worked

out in detail. All in all, these investigations are practical rather than theoretical in nature and will not be reviewed here. In the collateral field of demography, great strides have been made in the technique of deriving the many abbreviated tables essential to the adequate portrayal of the information buried in census data.

An innovation that has been introduced into certain recent mortality tables is an allowance for secular changes in the basic mortality rates. For many years there has been a steady improvement in mortality, which has cut down profit margins on annuities to a vanishing point. This has progressed at so rapid a rate that a mortality table has a fair chance of being obsolete before its publication date. To counteract this tendency, methods have been devised by which a projected increase in longevity can be "built in" to the mortality tables. Something of the sort was done in Britain in 1925; in 1949 Jenkins and Lew developed the theory in great detail in this country.

The concepts of mathematical statistics have been very sparingly applied to actuarial matters. This may seem surprising but it is true, particularly in this country. The educational program is putting more and more emphasis on mathematical statistics and it is to be expected that future developments will remedy this neglect. The χ^2 test of goodness of fit has from time to time cropped up in graduation, but it has never won general acceptance. The standard deviations of actuarial functions, commonplace in Europe, have been used in America only for about 25 years. Two problems in this connection still remain to be investigated. The first is how the mortality rate varies at a given age, that is, what is its frequency distribution? A satisfactory answer is not available because adequate data have never been tabulated in the necessary form and detail. The other problem arises because of the way life insurance companies make mortality investigations "by policies" and "by amounts" rather than "by lives." Since one "life," that is, one insured person, may have several policies, the variance of a mortality rate computed from "policies terminated by death" divided by "total policies," or from "actual death claims" divided by "insurance in force," will be vastly different from the variance of one computed in the basic manner. What is needed is a simple technique for developing conversion factors.

Finally, among the branches of mathematics which have been described in some detail in actuarial journals and considered as imminent or prospective tools for actuaries by their proponents are Fourier series, Stieltjes integrals, Tchebychef polynomials, binary calculation, Boolean algebra, and integral equations. There has even been an

attempt to explain mortality rates along the lines of the quantum theory. Most of these papers were written by Englishmen, and are perhaps a mark of that eccentricity which Henry Adams ascribed to Englishmen when he visited England in 1863. Still, when the next survey is made, some of these ideas doubtless, will prove to have fallen on fertile soil.

THE FUNDAMENTAL CONCEPTS

So much for a brief account of the improvements in mathematical techniques. There remains the examination of the fundamentals. What does actuarial science really deal with? Here we have a dearth of scientific papers, particularly in this country. There have been a few in Britain. Continental actuaries have devoted a great many words to the subject, but it cannot be said that they have shed much light on it. They have done a thorough job of working through classical life contingencies in the light of modern mathematical statistics, replacing probabilities by frequency distributions and expectations. Beyond this, their work has been largely academic, reflecting the university approach and the Ph.D. thesis rather than a basic endeavor to elucidate the fundamentals.

The fact is that very little has been done to elevate actuarial science to the level of a true science. F. M. Redington recognized this and summed up the state of affairs in discussing a paper before the British Institute in 1944. He believed that, although there had been considerable improvements in actuarial technique and craftsmanship during the last 50 years, there had been little or no real scientific advance. He did not believe, however, that the actuary's subject was entirely devoid of scientific content, or that actuaries were entirely devoid of scientific ability or curiosity. An examination of the *Journal* showed that almost all their scientific zest and ability were directed towards, and indeed dissipated in, theories of graduation and mortality, and it was clearly in that field that the obstacle must be sought. Conceptions such as 'underlying true rates of mortality,' 'standard deviation of deaths,' and even 'probability' itself, were in the nature of postulates rather than facts. It was with those shadowy but formidable postulates, rather than with the facts, that they were wrestling.

Although this appraisal was in the nature of a by-blow—the paper under discussion dealt with the validity of statistical tests of mortality tables— the uncomfortable truths are most pertinent. Many of the fields in which actuaries 'dissipated' their energies are no longer fertile. The preoccupation with mathematical mortality laws, so fashionable

at the beginning of the century, has to a large degree abated. Moreover, it is now quite generally recognized that these attempts to approximate mortality rates are an exercise in curve fitting rather than an attempt to phrase the underlying philosophy of mortality in mathematical terms. The theory of graduation and interpolation has probably reached its limit, in principle at least, with the development of the maximum and minimum technique. The earlier improvements were empirical in nature; now any problem is solved by standard mathematical methods once the objects to be accomplished have been formulated in mathematical terms.

The 'underlying true rates of mortality' referred to still present obstacles. As medicine has progressed both in its ability to understand and prevent disease, our concepts of mortality have been progressively changing. The notion of a single probability of death applicable to a specific age has become meaningless. Whether any part of it can be salvaged by working with several probabilities applicable to different degrees of impairment and to exposure to accidents remains to be seen. As new drugs are developed, yesterday's impairments become greatly modified in nature and incidence; as new methods of mass destruction are devised, tomorrow's deaths become more and more unpredictable.

So far, relatively crude procedures involving statistical frequencies and distributions are about the best that can be used. The stability of mortality phenomena, over short periods at least, has made these methods work. Note has been taken of the lack of adequate scientific studies of the statistical distribution of the mortality rate at each age. Investigations along these lines could well go hand in hand with the study of mortality from a medical point of view.

Once we know what mortality is all about and how it works at the grass-root level, we shall find no lack of mathematical techniques ready to hand for the elaboration of an actuarial science. Until the foundations are put on a scientific basis, future explorations can hardly yield more than small stones for the superstructure of the actuarial edifice, and actuarial science will develop more in the direction of an art than a science.